AD-A952 975

UNITED STATES NAVY

PROJECT SQUID

TECHNICAL MEMORANDUM No. CAL-14

THE CONSTRUCTION OF WAVE DIAGRAMS FOR THE STUDY OF ONE-DIMENSIONAL NON-STEADY GAS FLOW

By

GEORGE RUDINGER

and

LEONARD D. RINALDI

ATI-61546

10 MARCH 1948

CORNELL
AERONAUTICAL LABORATORY

83 11 15 263

Report No. DD-420-A-12
Contract No. N6ori-119

CORNELL AERONAUTICAL LABORATORY, INC.
BUFFALO 21, NEW YORK

SQUID TECHNICAL MEMORANDUM NO. CAL-14

THE CONSTRUCTION OF WAVE DIAGRAMS FOR THE STUDY OF ONE-DIMENSIONAL NON-STEADY GAS FLOW

BY

GEORGE RUDINGER

AND

LEONARD D. RINALDI

10 MARCH 1948

CORNELL AERONAUTICAL LABORATORY
Buffalo, New York

Report DD-420-A-12

TABLE OF CONTENTS

CORNELL AERONAUTICAL LABORATORY
Buffalo, New York

Report DD-420-A-12

ABSTRACT

This paper is an introduction to the method of characteristics for solving problems of one-dimensional, non-steady gas flow. By following Riemann's approach, it can be shown that wave elements propagate along certain lines which are identical with characteristics lines. Construction of these lines in the position-time plane gives a clear picture of the motion of waves in ducts. Several new techniques for obtaining solutions of special problems are suggested.

CORNELL AERONAUTICAL LABORATORY
Buffalo, New York

Report DD-420-A-12

INTRODUCTION

Problems in the field of non-steady compressible flow are becoming increasingly important. As long as the flow may be treated as one-dimensional, the method of characteristics is a most useful tool to obtain graphical solutions to given problems. One objection to this method occasionally raised is that "it is a mechanical procedure where the physical picture may be completely lost". This has not been the experience of the writers, who feel that keeping the physical side in mind makes application of the method considerably easier. This difference of view may, perhaps, be explained by pointing out two slightly different interpretations of the method.

Mathematically, the problem is to solve certain partial differential equations. These equations have associated with them two families of curves, the characteristics. Through every point in the position-time plane two such lines may be drawn. The network of these lines forming the so-called characteristics diagram, covers the entire plane and its pattern is determined by the boundary conditions. A change of conditions at some region of the boundary affects the diagram only along characteristics lines starting at that region. In any given case, the characteristics diagram is constructed according to certain rules and the values of flow velocity and state parameters are obtained from the diagram.

The other possibility is to derive the propagation laws for wave elements directly from the fundamental relations. It can be shown that these wave elements travel along certain paths in the position-time plane which are identical with characteristic lines. Diagrams constructed in this manner will be called wave diagrams. Values of flow velocities and state parameters are obtained from the diagram in the same way as from a characteristics diagram and, the result, of course, is the same in both cases. Both procedures are used in the literature[1,2,3,4,5,6].

In either case, the method allows investigation of non-steady, one-dimensional flow, giving flow velocities and state parameters as functions of time and position. It is possible to treat flows in ducts of variable cross-section provided they may be considered one-dimensional with sufficient accuracy. Experiments have shown that in certain cases the flow pattern in the immediate vicinity of an open end may not be considered one-dimensional. Therefore, at such points, velocity determinations may not lead to significant results. This report being of an introductory nature, is limited to isentropic flow and, therefore, does not treat heat addition or very strong shock waves. The method may be extended to cover such cases but the procedures then become considerably more complicated.

Although the principles of the method presented in this paper are not new, the need was felt for a description stressing more the practical side than was done in previous presentations. No claim for originality is made but it is believed that some of the procedures described are new.

The method as described in this paper has been found to be a useful tool for preliminary investigations of problems of non-steady gas flow since a general picture of the wave motion is readily obtained. For a more refined treatment it is necessary that more elaborate procedures be used which include non-isentropic flow and continuous changes of cross section. These techniques will be reviewed in a future paper.

LIST OF SYMBOLS

u		Particle velocity
a		Velocity of sound
w		Velocity of wave propagation
p		Pressure
ρ		Density
x		Distance coordinate
t		Time
a_o		Arbitrary reference value of a
p_o		Arbitrary reference value of p
L_o		Arbitrary reference value of length
ξ	=	$\frac{x}{L_o}$
τ	=	$\frac{a_o t}{L_o}$
U	=	$\frac{u}{a_o}$
A	=	$\frac{a}{a_o}$
W	=	$\frac{w}{a_o}$
A_e	=	Velocity of sound outside the tube (non-dimensional)
A^*	=	$A^2 + \frac{1}{\beta} U^2$
P	=	$U + \beta A$
Q	=	$U - \beta A$
ΔA		Change of A across a wave element

ΔU	Change of U across a wave element
K	Function defined by equation (18)
S	Cross-sectional area of tube
V	Piston velocity (non-dimensional)
γ	Ratio of specific heats
β	$= \dfrac{2}{\gamma - 1}$
λ	$= \dfrac{\gamma + 1}{\gamma - 1}$

I. PRESSURE WAVES OF FINITE AMPLITUDE

A problem in non-steady gas flow is solved once the state parameters and flow velocity are known at any point as functions of time. All studies in this paper relate to one-dimensional flow and isentropic state transformations. It is sufficient to find the solution for the flow velocity u and one state parameter since other state parameters may then be obtained by means of the isentropic flow relations. Instead of pressure p or density ρ, it is more convenient to select the velocity of sound a as the state parameter where $a = \sqrt{\gamma \frac{p}{\rho}}$ and γ is the ratio of specific heats. The derivation of the relations leading to the construction of wave diagrams follows the general approach of Riemann[7]. In the Appendix, an outline is given for the mathematical basis of the method of characteristics as applied to non-steady, one-dimensional flow.

The motion of the gas is governed[8] by the equation of continuity (1), the equation of motion (2) and the condition of isentropic flow (3)

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{a^2}{\rho}\frac{\partial \rho}{\partial x} \tag{2}$$

$$\frac{da}{a} = \frac{\gamma-1}{2}\frac{d\rho}{\rho} \tag{3}$$

Multiplying equation (1) by $\pm\frac{a}{\rho}$ and adding equations (1) and (2) gives

$$\frac{\partial u}{\partial t} \pm \frac{a}{\rho}\frac{\partial \rho}{\partial t} + (u \pm a)\left(\frac{\partial u}{\partial x} \pm \frac{a}{\rho}\frac{\partial \rho}{\partial x}\right) = 0$$

By using equation (3), ρ may be eliminated from this equation, yielding-ing

$$\frac{\partial}{\partial t}\left(u \pm \frac{2}{\gamma-1}a\right) + (u \pm a)\cdot\frac{\partial}{\partial x}\left(u \pm \frac{2}{\gamma-1}a\right) = 0 \tag{4}$$

It is convenient at this point to introduce the dimensionless variables

$$U = \frac{u}{a_o}\ ,\quad A = \frac{a}{a_o}\ ,\quad \xi = \frac{x}{L_o}\ \text{and}\quad \tau = \frac{a_o t}{L_o}$$

(positive to the right) ~~and~~

where a_o and L_o are arbitrary reference values of the velocity of sound and length, respectively. In dimensionless form, equation (4) becomes

$$\frac{\partial}{\partial \tau}\left(U \pm \frac{2}{\gamma-1}A\right) + (U \pm A)\frac{\partial}{\partial \xi}\left(U \pm \frac{2}{\gamma-1}A\right) = 0$$

Defining

$$P = U + \beta A$$

$$Q = U - \beta A$$

and

$$\beta = \frac{2}{\gamma - 1}$$

the last relation represents the two equations

$$\frac{\partial P}{\partial \tau} + (U + A) \frac{\partial P}{\partial \xi} = 0 \qquad (4')$$

and

$$\frac{\partial Q}{\partial \tau} + (U - A) \frac{\partial Q}{\partial \xi} = 0 \qquad (4'')$$

Equations (4) are wave equations which indicate that the quantities P and Q propagate with velocities $W = U + A$ and $W = U - A$, respectively. Here, $W = \frac{w}{a_0}$ is the dimensionless form of the propagation velocity w. As long as U is smaller than A, it is apparent that equation (4') represents waves travelling from left to right while equation (4'') epresents waves travelling from right to left. In general, waves travelling in both directions are present at the same time. According to equation (4'), a value of P remains unchanged for an observer who moves with the wave from left to right with a velocity U + A, whether or not he crosses waves travelling from right to left. However, depending on how the wave was created, each part of it has its own value of P propagating with its own characteristic velocity U + A. Similarly, values of Q remain constant for an observer moving with a velocity U - A, whether or not he crosses waves travelling from left to right. Therefore, if there are no waves travelling from left to right in a certain region, P is constant within that region. Similarly, Q is constant within a region where there are no waves travelling from right to left.

In the absence of a general solution for equations (4') and (4"), the following procedure is used to solve given problems. Each wave is approximated by a number of wave elements in the form of steps each of which is characterized by its values of A and U (Fig. 1). The values of U and A change from one element to the next and the increments will be denoted by ΔU and ΔA,

$$\Delta U = U_i - U_{i-1}$$

$$\Delta A = A_i - A_{i-1} \qquad (5)$$

In the ξ, τ -plane, the path of each element may be drawn as a line, the inclination of which corresponds to the wave velocity $\frac{d\xi}{d\tau} = W$ (Fig. 2). Each line is labelled by its strength ΔA. The value of ΔU is not required because of the relation between ΔA and ΔU to be derived below. These lines divide the field into a number of regions which for easy identification will be numbered 1, 2, 3, etc. Such plots will be referred to as wave diagrams. All lines in Fig. 2 represent elements of a wave travelling from left to right and, therefore, all regions have the same value of Q because of the absence of waves moving from right to left. It then follows from the definition of Q that

$$\Delta U = \beta \Delta A \tag{6'}$$

This relates the increments of U and A across a wave travelling from left to right. However, each element of this wave has a different value of P

$$P_1 = U_1 + \beta A_1\,, \quad P_2 = U_2 + \beta A_2\,, \quad P_3 = U_3 + \beta A_3\,, \ldots\ldots$$

and the propagation velocity $W = U + A$ is different for each element.

Similar relations hold for the case in which only waves travelling from right to left are present. Then, P is constant and Q has different values for each element of the wave. The definition of P yields

$$\Delta U = -\beta \Delta A \tag{6''}$$

which relates the increments of U and A across a wave travelling from right to left.

In order to study the intersection of two waves travelling in opposite directions, consider one element of each having a strength ΔA and ΔA_1, respectively, before intersection (Fig. 3). After intersection, let them assume the values $\Delta A'$ and $\Delta A_1'$. Regions 1 and 3 are separated by a wave travelling from right to left across which P is constant, therefore, $P_1 = P_3$. Similarly, $P_2 = P_4$, $Q_1 = Q_2$, and $Q_3 = Q_4$.

From these conditions, it follows that $\Delta A = \Delta A'$ and $\Delta A_1 = \Delta A_1'$. This means that the strength of a wave element is unaltered by intersection with other elements. However, its speed is changed by intersection. For example, the wave marked ΔA in Fig. 3 moves with a speed $W_1 = U_1 + A_1$ while after intersection its velocity is given by

$$W_3 = (U_1 - \beta \Delta A_1) + (A_1 + \Delta A_1) = U_1 + A_1 + (1-\beta) \Delta A_1$$

A positive value of ΔA corresponds to a wave for which the velocity of sound, or the density, or the pressure after passage of the wave is higher than before. This is called a compression wave. Similarly, a negative value of ΔA corresponds to an expansion wave.

II PROPAGATION VELOCITY OF COMPRESSION WAVES

It was shown in Section I that wave elements travel with a velocity $w = u \pm a$. Imagine that small disturbances are created in some manner and propagate through the gas in a tube. The velocity of sound at a point after the first element of the wave has passed will be higher or lower depending upon whether the wave is a compression or an expansion wave. In the case of a compression wave, each wave element travels in a medium which has been compressed by the previous elements and will, therefore, move with a velocity which is greater than that of the preceding one. Successive wave elements may, therefore, overtake each other forming a single element of combined strength (shock wave).

In the case of an expansion wave, each element effects a pressure drop and, the following waves travel slower. Therefore, successive elements fan out and no shock is formed.

If a shock wave is formed, it is known that the flow is no longer isentropic and the state parameters on both sides of the shock, and the propagation velocity are then related by the Rankine-Hugoniot equations[8,9]. It, therefore, appears that the method described in Section I breaks down as soon as shock waves occur, that is when compression wave elements combine. However, for shocks of moderate strength, the isentropic flow relations give values of the state parameters that agree very closely with those obtained from the Rankine-Hugoniot equations. On the other hand, the deviation of the propagation velocity of shock waves from the value $u \pm a$ obtained for isentropic flow is appreciable even for weak shocks. It is possible to include shock waves of moderate strength in the wave diagram by still treating changes of state across the wave as isentropic (equations 6) but using the relation for the propagation velocity of shock waves which is given by

$$w = u_1 \pm \sqrt{\frac{\rho_2}{\rho_1} \cdot \frac{p_2 - p_1}{\rho_2 - \rho_1}} \qquad (7)$$

The subscripts 1 and 2 refer to the conditions on both sides of the shock, and the $\pm$ sign indicates the direction of motion. Introducing the isentropic relations, equation (7) may be written as

$$W = U_1 \pm A_1 \sqrt{\frac{\left(\frac{A_2}{A_1}\right)^{\beta} \cdot \left[\left(\frac{A_2}{A_1}\right)^{\beta\gamma} - 1\right]}{\gamma \cdot \left[\left(\frac{A_2}{A_1}\right)^{\beta} - 1\right]}}$$

which is in the non-dimensional form used in Section I, and again with $\beta = \frac{2}{\gamma - 1}$. Writing $A_2 = A_1 + \Delta A$ this becomes

$$W = U_1 \pm A_1 \sqrt{\frac{\left(1 + \frac{\Delta A}{A_1}\right)^{\beta} \cdot \left[\left(1 + \frac{\Delta A}{A_1}\right)^{\beta\gamma} - 1\right]}{\gamma \cdot \left[\left(1 + \frac{\Delta A}{A_1}\right)^{\beta} - 1\right]}}$$

which may be expanded into a series. Retaining only the first three terms leads to

$$W = U_1 \pm A_1 \left[1 + \frac{\lambda}{2} \cdot \frac{\Delta A}{A_1} + \left(\frac{\lambda^2}{8} - \frac{\lambda}{6}\right) \cdot \left(\frac{\Delta A}{A_1}\right)^2\right] \tag{8}$$

where $\lambda = \frac{\gamma + 1}{\gamma - 1}$

Since even weak elements of a compression wave may be considered as a combination of still weaker ones, equation (8) should be used for all compression wave elements. It is obvious that this equation approaches the previous relation $W = U \pm A$ for very weak waves ($\Delta A \rightarrow 0$). In many cases the third term of the series may be neglected.

For increasing strength of the shock waves, the assumption of isentropic changes of state leads to errors which eventually will exceed the limits of the desired accuracy. In general, shock waves with values of ΔA up to 0.14, corresponding to a pressure ratio $\frac{p_2}{p_1} = 2.5$ (for $\gamma = 1.4$) may be treated in the described way. For stronger waves, where the isentropic relations do not hold with sufficient accuracy, methods have also been developed[4] but these will not be treated in this paper.

As pointed out before, expansion waves do not form shocks and the relation for the propagation velocity $W = U \pm A$ derived in Section I should be used throughout.

III. BOUNDARY CONDITIONS AND WAVE REFLECTIONS

Once the initial conditions are given, the methods described in Sections I and II allow determination of the state of the gas at any point or any time inside a tube of constant cross section.

When the waves reach a discontinuity, such as the end of a tube or a change of cross section, wave reflections must occur in order to satisfy the boundary conditions.

1. <u>Tube closed by a moving piston</u>

If a tube is closed by a piston moving with a velocity V, the gas adjacent to the piston must move with the same velocity so that at the boundary $U = V$. Fig. 4 shows a wave, which may be a compression or an expansion wave, being reflected from a piston. In region 1 ahead of the wave $U_1 = V$. From equations (5) and (6) one obtains for region 2

$$A_2 = A_1 + \Delta A$$

and

$$U_2 = V + \beta \Delta A$$

Denoting the strength of the reflected wave by $\Delta A'$ then for region 3

$$A_3 = A_2 + \Delta A'$$

and

$$U_3 = V = U_2 - \beta \Delta A' = V + \beta \Delta A - \beta \Delta A'$$

and, therefore

$$\Delta A = \Delta A' \qquad (9)$$

This shows that the reflected wave has the same strength and sign as the incident wave.

2. <u>Closed end of a tube</u>

This is merely a special case of a moving piston with $V = 0$, so that the boundary condition becomes $U = 0$ and for the reflected wave the same relation $\Delta A = \Delta A'$ applies.

3. Open end of a tube

At an open end of a tube several cases of wave reflection are possible.

The conditions before arrival of a wave element are known but in some cases it is not possible to anticipate the type of flow after reflection. In such cases, one can only assume a certain flow and then check whether the results are in agreement with the assumption.

In order to calculate wave reflections at an open end the velocity of sound there must be given. If the state of the outflowing gas is isentropically related to conditions outside the tube, then the corresponding velocity of sound at the end is known. If this condition is not fulfilled the velocity of sound at the open end is not known and some reasonable value has to be assumed. The velocity of sound at the open end, in non-dimensional form, will be denoted by A_e. As an example, consider the outflow of gas from a container through a tube to the open atmosphere. If the pressure in the container was reached by isentropic compression from atmospheric pressure, then A_e is equal to the velocity of sound in the open atmosphere. If the pressure was reached in any other way, an assumption for A_e has to be made, for instance the value of the sound velocity obtained if the gas were expanded isentropically from container to atmospheric pressure. All relations will be derived with the aid of Fig. 4. With configurations which do not allow the flow to become supersonic at the open end only two types of outflow are possible: (a) $U_3 < A_3$ and (b) $U_3 = A_3$.

(a) Outflow with $\pm U_3 < A_3$ *

As long as the outflow velocity at the open end is less than the local velocity of sound, the pressure is equal to that prevailing in the external medium. Since at that pressure the velocity of sound is equal to A_e, the wave must be reflected so that the boundary condition $A_3 = A_e$ (Fig. 4) is maintained. The following relations may be written.

$$A_2 = A_1 + \Delta A$$

and $$A_3 = A_e = A_2 + \Delta A' = A_1 + \Delta A + \Delta A'$$

Therefore, $$\Delta A' = -\Delta A + A_e - A_1 \tag{10}$$

If $A_1 = A_e$ which is the most common case, a wave is reflected with unaltered strength but a compression wave is changed into an expansion wave and vice versa. Once $\Delta A'$ has been found, U_3 may be calculated and checked for the condition that the outflow velocity does not exceed A_3.

---

*The upper sign for U_3 refers to the right end of the tube, and the lower to the left end since there $U_3 < 0$ for outflow.

(b) Outflow with $\pm U_3 = A_3$

As before,

$$A_2 = A_1 + \Delta A$$

$$U_2 = U_1 \pm \beta \Delta A$$

and $A_3 = A_2 + \Delta A' = A_1 + \Delta A + \Delta A'$

$$U_3 = U_2 \mp \beta \Delta A' = U_1 \pm \beta (\Delta A - \Delta A')$$

The boundary condition $\pm U_3 = A_3$ leads to $\pm U_1 + \beta (\Delta A - \Delta A') = A_1 + \Delta A + \Delta A'$ and, therefore,

$$\Delta A' = \frac{(\beta - 1) \Delta A \pm U_1 - A_1}{\beta + 1} \tag{11}$$

where the upper sign refers to the right end and the lower to the left end of the tube.

(c) Inflow

A pressure gradient is required to produce inflow and, therefore the pressure at the open end must be less than the outside pressure. This pressure drop may be calculated from the energy equation[8,9] which may be written as

$$A_3^2 + \frac{1}{\beta} U_3^2 = A_e^2 \tag{12}$$

In addition, the strength of the reflected wave must be such that the increments for U and A satisfy equations (5) and (6).

$$U_3 - U_2 = \mp \beta (A_3 - A_2)$$

where, again, the upper sign refers to the right end. The boundary condition (12) and the last relation may be combined and solved for A_3 and U_3.

Considering the right end only, the solution of the quadratic equation for U_3 becomes

$$U_3 = \frac{(\beta A_2 + U_2) \pm \beta \sqrt{(1+\beta) A_e^2 - \frac{1}{\beta}(\beta A_2 + U_2)^2}}{\beta + 1}$$

Only one of the two solutions has physical significance. Since the first term in the numerator is always positive, and, for inflow at the right end, U_3 must be negative, the minus sign in front of the root must be used. Similar considerations may be applied to the left end.

For computing purposes, the most convenient method is to first solve for A_3 which leads to

$$A_3 = \frac{\pm (U_2 \pm \beta A_2) \pm \sqrt{(\beta + 1) A_e^2 - \frac{(U_2 \pm \beta A_2)^2}{\beta}}}{\beta + 1} \, . \qquad (13)$$

The next step is to calculate the strength of the reflected wave

$$\Delta A' = A_3 - A_2 \qquad (14)$$

and finally

$$U_3 = U_2 \mp \beta \, \Delta A' \qquad (15)$$

In equations (13) and (15) the upper sign always refers to the right end and the lower to the left end of the tube. As a check, the result for U_3 must correspond to inflow.

4. Change of cross section

If the cross section of the tube varies, the state of the flowing gas also changes. Problems of this type may be treated in two different ways. First, it is possible to include continuous changes of cross section in the fundamental continuity equation (1) and then derive the relations governing the characteristics network[5,6]. This is a direct approach but the resulting procedures become rather complicated.

Second, continuous changes of cross section may be approximated by discrete steps as was done for the flow parameters. In this way, the construction of the wave diagram remains unaltered, and only certain boundary conditions at those points where the cross section changes have to be taken into account.

When a wave reaches a change of cross section, it will be partly reflected and partly transmitted. The strengths of the incident, reflected and transmitted waves will be denoted by ΔA, $\Delta A'$ and $\Delta A''$, respectively.

The symbol S' will be used for the cross sectional area of that part of the duct in which the incident and reflected waves travel, while S'' will be written for the cross sectional area of the region through which the transmitted wave travels (Fig. 5).

There is steady flow connecting regions 1 and 4 and also 3 and 5. The two conditions which must be fulfilled in each case are continuity of mass flow and conservation of energy. These may be written as

$$S' \cdot A_1^{\beta} \cdot U_1 = S'' \cdot A_4^{\beta} \cdot U_4 \qquad (16)$$

and

$$U_1^2 + \beta A_1^2 = U_4^2 + \beta A_4^2 \qquad (17)$$

and equivalent relations for regions 3 and 5.

The problem to be solved is the following: given A_1, U_1 and A, find A_2, U_2, A_3, U_3, A_4, U_4, A_5, U_5, $\Delta A'$ and $\Delta A''$. These are ten unknowns and the same number of relations is, therefore, required. Going from region 1 to region 2, equations (5) and (6) give two relations. Similarly, four additional equations are obtained by going from 2 to 3 and from 4 to 5. Equations (16) and (17) and the equivalent relations connecting regions 3 and 5 complete the system of equations. Analytical solution of the system is very tedious and several graphical methods have been described in the literature [1,5]. A different numerical approach, however, has been found to be more satisfactory.

Equations (16) and (17) contain U_4 and A_4 as the only unknowns. The following symbol is introduced

$$A^{*2} = A^2 + \frac{1}{\beta} U^2$$

so that equation (17) becomes

$$A_1^* = A_4^* \qquad (17')$$

From equations (17) and (17')

$$U_1^2 = \beta A_1^{*2} - \beta A_1^2$$

and

$$U_4^2 = \beta A_4^{*2} - \beta A_4^2 .$$

If these are introduced into equation (16), the quantities may be rearranged to

$$\left(\frac{A_4}{A_4^*}\right)^{2\beta} \cdot \left[1 - \left(\frac{A_4}{A_4^*}\right)^2\right] = \left(\frac{A_1}{A_1^*}\right)^{2\beta} \cdot \left[1 - \left(\frac{A_1}{A_1^*}\right)^2\right] \cdot \left(\frac{S'}{S''}\right)^2$$

and with the new function

$$K = \left(\frac{A}{A^*}\right)^{2\beta} \cdot \left[1 - \left(\frac{A}{A^*}\right)^2\right] \tag{18}$$

equation (16) becomes

$$K_4 = K_1 \cdot \left(\frac{S'}{S''}\right)^2 \tag{19}$$

In order to evaluate K from equation (18), the value of A/A^* must be known. From the definition of A^*

$$\frac{A}{A^*} = \frac{1}{1 + \frac{1}{\beta}\left(\frac{U}{A}\right)^2} \tag{20}$$

Figure 6 shows a plot of K as function of A/A^* where the progress of the calculation is also indicated. First, compute A_1/A_1^* from equation (20) and find K_1 and K_4 from the graph and equation (19). K_4 then corresponds to a value of A_4/A_4^* and since $A_4^* = A_1^*$, one obtains A_4 and, again from equation (20), also U_4. One might think that for certain values of

$\frac{S''}{S'}$ the value of K_4 could exceed the possible maximum (Fig. 6). This case, however, will not occur because as is easily shown, the maximum value of K corresponds to a Mach number of one which may only occur at the narrowest section of a duct. Since supersonic flows will not be considered here, only the right branch of the curve is important. For practical purposes, numerical values of K are given in Table 1 for $\gamma = 1.4$.

The relations connecting the U and A values in regions 3 and 5 are the same as for regions 1 and 4 but none of the values are known. If $\Delta A''$ were known, the computation could proceed in the same way as above and finally $\Delta A'$ would be found by applying equations (5) and (6) to regions 2 and 3. The values for region 2 are calculated in the usual manner from those in region 1 and ΔA. The analytical approach to the problem leads to very cumbersome relations. It was found more convenient to guess the value of $\Delta A''$, and then calculate the values for regions 3 and 5 in the same way as for regions 1 and 4. Equations (5) and (6) will give two values of $\Delta A'$ from the difference of the U and A values respectively, and these two values must agree within the accuracy desired. If they do not agree, the procedure is repeated with a new guess of $\Delta A''$.*

---

* A number of cases were calculated and it was found that the strength of the transmitted and reflected waves does not depend very much on the value of A_1. From these results a graph was prepared to be used as an aid for making the first guess of $\Delta A''$. This is shown in Fig. 7 where the ratio $\Delta A''/\Delta A$ is plotted as function of S''/S' for different values of U_1.

5. Interface between two gases

In all cases treated so far, one gas only was involved in the flow and all changes of state were isentropic. The problem will now be generalized to the flow of two gases separated by a plane interface. The necessary condition for the interface to exist is that both gas velocity and gas pressure on each side be equal. Changes of state within each gas will still be treated as isentropic. A wave element arriving at the interface will be partly transmitted and partly reflected, and at the same time the velocity of the interface will be changed.

In Fig. 8, the position of the interface is indicated by a dotted line. Interface, incident, reflected and transmitted waves divide the area into five regions marked 1 to 5. In the analysis of the wave reflections it is convenient to use actual state parameters and to introduce the usual non-dimensional variables in the final formulas only. With reference to Fig. 8, the boundary conditions may be written

$$u_1 = u_4 \quad , \quad p_1 = p_4 ,$$
$$\text{and} \quad u_3 = u_5 \quad , \quad p_3 = p_5 . \tag{21}$$

One may imagine the gases in regions 1 and 4 to be isentropically compressed from some arbitrary pressure p_0 at which the corresponding velocities of sound are a_{10} and a_{40}, respectively,

$$\frac{p_1}{p_o} = \left(\frac{a_1}{a_{10}}\right)^{\frac{2\gamma'}{\gamma'-1}}$$

and

$$\frac{p_4}{p_o} = \left(\frac{a_4}{a_{40}}\right)^{\frac{2\gamma''}{\gamma''-1}}$$

where the ratio of specific heats γ' refers to the gas in which the reflected wave travels while γ'' refers to the gas on the other side of the interface.

The pressures in regions 3 and 5 may also be expressed in terms of the sound velocities taking into account the strength of the wave elements,

$$\frac{p_3}{p_o} = \left(\frac{a_1 + \Delta a + \Delta a'}{a_{10}}\right)^{\frac{2\gamma'}{\gamma'-1}}$$

and

$$\frac{p_5}{p_o} = \left(\frac{a_4 + \Delta a''}{a_{40}}\right)^{\frac{2\gamma''}{\gamma''-1}}$$

If the incident pressure wave travels from left to right one may write for the velocities

$$u_3 = u_1 + \frac{2}{\gamma'-1}\,(\Delta a - \Delta a')$$

and

$$u_5 = u_4 + \frac{2}{\gamma''-1}\,\Delta a''.$$

Applying the boundary conditions (21) the pressure and velocity relations lead to the following two equations for $\Delta a'$ and $\Delta a''$

$$\frac{2}{\gamma''-1}\,\Delta a'' + \frac{2}{\gamma'-1}\,\Delta a' = \frac{2}{\gamma'-1}\,\Delta a$$

$$\left(\frac{a_1 + \Delta a + \Delta a'}{a_1}\right)^{\frac{2\gamma'}{\gamma'-1}} = \left(\frac{a_4 + \Delta a''}{a_4}\right)^{\frac{2\gamma''}{\gamma''-1}} \tag{22}$$

In this general form, the equations can be solved only after numerical values have been introduced.

Since, generally, the wave strength ΔA is very small compared to the sound velocity, the second equation (22) may be expanded into a series of which the linear terms only are retained. It is then possible to solve the equations with the results

$$\frac{\Delta A'}{\Delta A} = \frac{A_1\,\gamma'' - A_4\,\gamma'}{A_1\,\gamma'' + A_4\,\gamma'}$$

and
$$\frac{\Delta A''}{\Delta A} = \frac{2\,A_4\,\gamma''}{A_1\,\gamma'' + A_4\,\gamma'} \cdot \frac{\gamma''-1}{\gamma'-1} \tag{22'}$$

where the values have been made dimensionless by dividing them by an arbitrary reference velocity of sound. Although the above relations were derived for a wave travelling from left to right they apply in unchanged form to waves travelling from right to left.

The above treatment includes the special case of two masses of the same gas but of different temperatures moving together. Thus γ' equals γ'' and equations (22') reduce to

$$\frac{\Delta A'}{\Delta A} = \frac{A_1 - A_4}{A_1 + A_4} \quad \text{and} \quad \frac{\Delta A''}{\Delta A} = \frac{2A_4}{A_1 + A_4} \tag{22''}$$

IV PHYSICAL CONDITIONS

A wave diagram can be drawn for specific cases only when the physical phenomena which create non-steady flow are known. Most problems met in practice will be covered by one of the cases described below or by a combination of some of them. Some consideration is also given to the treatment of strong expansion waves since expansion shock waves cannot exist.

1. Moving piston

It was noted in Section III that for a tube closed by a piston moving with a velocity V, the gas adjacent to the piston must move with the same velocity so that at the boundary $V = U$. A sudden change of V will cause a wave to be emitted from the piston so that $\Delta U = \Delta V$ and the strength of the wave is given by

$$\Delta A = \pm \frac{1}{\beta} \Delta U$$

in accordance with equations (6). If V varies continuously with time, suitable discontinuous steps ΔV must be selected to approximate the motion of the piston. At the instant of each step, a wave of corresponding strength ΔA is emitted.

2. Sudden removal of a partition

A very general type of non-steady flow is produced by the sudden removal of a partition separating two gases at different pressures and temperatures. This creates an interface which is assumed to remain plane. The boundary condition which must then be fulfilled is that pressures and velocities be equal on both sides of the interface. It is clear that the removal of the partition must create an expansion wave which advances into the region of higher pressure and a compression wave which moves into the region of lower pressure. The strengths of these two waves will be derived below. Subscripts refer to the regions as shown in Fig. 9.

The condition in regions 1 and 2 are given and assuming that there is no heat exchange across the interface, the isentropic relations may be applied to each gas individually, as follows,

$$\frac{p_1}{p_3} = \left(\frac{a_1}{a_3}\right)^{\frac{2\gamma'}{\gamma'-1}}$$

and

$$\frac{p_2}{p_4} = \left(\frac{a_2}{a_4}\right)^{\frac{2\gamma''}{\gamma''-1}}$$

where the ratios of specific heats are γ' and γ'', respectively. The additional boundary condition is

$$p_3 = p_4.$$

Introducing the strength of the waves, these equations may be combined to give a relation between Δa_1 and Δa_2

$$p_1 \cdot \left(\frac{a_1 + \Delta a_1}{a_1}\right)^{\frac{2\gamma'}{\gamma'-1}} = p_2 \cdot \left(\frac{a_2 + \Delta a_2}{a_2}\right)^{\frac{2\gamma''}{\gamma''-1}} \qquad (23)$$

The velocity relations (6) give

$$u_3 = u_1 - \beta' \Delta a_1$$

and $$u_4 = u_2 + \beta'' \Delta a_2 .$$

With the boundary conditions

$$u_1 = u_2$$

and $$u_3 = u_4,$$

these may be combined to

$$\beta' \Delta a_1 = - \beta'' \Delta a_2 \tag{24}$$

For given numerical values equations (23) and (24) may be solved for Δa_1 and Δa_2. In the special case of $\gamma' = \gamma''$ where, however, the two gases need not be the same, one also has $\beta' = \beta''$ and from equation (24) $\Delta a_1 = - \Delta a_2$. The actual value of Δa may then be found from equation (23).

One may specialize the problem still further by assuming the same gas on both sides of the partition and isentropic relation between the states 1 and 2. Then

$$\frac{p_1}{p_2} = \left(\frac{a_1}{a_2} \right)^{\frac{2\gamma}{\gamma - 1}}$$

It is obvious that the states of the gas in regions 3 and 4 must become identical since they are also related to the original states by the isentropic law and must both have the same pressure. In this special case, an interface is not formed and equation (23) simplifies to

$$\Delta a_2 = - \Delta a_1 = \frac{a_1 - a_2}{2}$$

If an interface is formed at an open end of a tube, the two waves are still calculated as before. The one directed outward is immediately reflected as described in Section III and combines with the other one to form a single wave advancing into the tube.

3. Heat addition to gas flow

If heat is added to the flowing gas the original assumption of isentropic flow does not hold any longer and equation (3) must be replaced by the energy equation. The graphical procedures then become considerably more complicated and will not be treated here[6].

At times, it may be possible to overcome the difficulties introduced by heat addition by making simplifying assumptions about the process[10]. For instance, combustion may be assumed to take place instantaneously. If this is done, an instantaneous change of state of the gas takes place in a given volume and this case then becomes identical with the one of a suddenly removed partition treated before. The assumption of instantaneous combustion may be justified for high rates of burning where constant volume combustion is approached. For low rates of burning, serious errors might be introduced by this method. It may be possible, however, to treat these problems by making certain simplifying assumptions about the combustion process. Such procedures are being investigated at the present time.

4. Strong expansion waves

It was pointed out in Section I that continuous changes of state must be approximated by small discontinuous steps. These correspond to some arbitrarily selected value of ΔA which then determines the accuracy of the approximation. While larger steps than these have physical significance in the case of compression waves developing into a shock wave, larger values could not appear in expansion waves since expansion shock waves do not occur. In Section III the strength of reflected and transmitted waves under various conditions was determined but the actual values of ΔA were not considered. Often the calculated value of ΔA corresponds to an expansion wave stronger than the arbitrary step size selected for the approximation of continuous changes. This case will occur, for instance, when a shock wave is reflected as an expansion wave from the open end of a tube. For such cases, the expansion wave has to be split up into elements of a strength not exceeding the step size selected. These elements will all start from the same point of the wave diagram and then fan out. It makes little difference whether all elements are made of equal strength or not, as long as they do not exceed the selected step size.

V. PRACTICAL APPLICATIONS

1. General remarks

It is now possible to construct wave diagrams for the solution of specific problems. Two examples were selected. The first of these considers the gas oscillations created in a tube by a prescribed motion of a piston. The other treats the case of a wave reflected at a change of cross-section.

Before starting to draw a wave diagram, one must decide on the size of steps into which continuous changes of state will be broken down. Selecting a value for ΔA fixes the steps for all other variables because of equations (6) and of the assumed isentropic changes of state. In general, a value of ΔA equal to 0.020 will give satisfactory results and this value will be used in the following examples. For better approximations a smaller value may be required while for a qualitative investigation of a process one might choose larger values.

It is customary to draw compression waves as solid lines and expansion waves as broken lines, and to label each line with its value of ΔA. The wave velocity $W = \frac{d\xi}{d\tau}$ is the slope of the lines. The regions formed by intersecting lines are numbered for convenience and a record of the corresponding values of A, U and other quantities of interest are tabulated on a separate sheet.

2. Gas waves produced by a moving piston

Assume a piston moving in a tube which is open at one end as shown in Fig. 10. All values have been made dimensionless, using the length of the tube and the velocity of sound outside the tube as reference values. The continuous motion of the piston is shown by the full line and approximated by the dotted line, as follows

$$\begin{aligned}
&\tau < 0\,, && \xi = 0\,, && V = 0\,,\\
&0<\tau<0.5\,, && \xi = 0.2\tau\,, && V = 0.2,\\
&0.5<\tau<1\,, && \xi = 0.1\,, && V = 0,\\
&1<\tau<1.5\,, && \xi = 0.3 - 0.2\tau\,, && V = -0.2,\\
&\tau > 1.5\,, && \xi = 0\,, && V = 0\,.
\end{aligned}$$

This example was chosen not for its practical value but rather to include as many procedures as possible.

When τ is zero, the piston velocity suddenly changes from zero to 0.2 and since the change of the gas velocity adjacent to the piston must be equal to this, a wave must be emitted of such strength that it corresponds to $\Delta U = 0.2$. Thus from equation (6') $\Delta A = 0.04$ since for air $\gamma = 1.4$ and $\beta = \frac{2}{\gamma - 1} = 5$. The velocity of this wave is 1.126, as given by equation (8). At $\tau = 0.5$, the piston velocity changes to zero which corresponds to the emission of an expansion wave of $\Delta A = -0.04$. Since the step size selected was 0.02 this expansion wave must be split up into two elements of $\Delta A = -0.02$ each.

Being expansion waves, they travel with a velocity (U + A) in their respective regions (Table 2). In the same way, two more expansion waves are emitted at $\tau = 1$ and another compression line $\Delta A = 0.04$ starts from $\tau = 1.5$.

When the first compression wave reaches the open end, it is reflected as an expansion wave but must again be split up into small elements because of its strength. These reflected waves intersect the consecutive oncoming waves and, each time two lines intersect, their velocity after intersection must be calculated according to the rules in Sections I and II. Reference to Fig. 10 and Table 2 will make it easy to follow the procedure. When the waves return to the piston, they are then reflected again so as to satisfy the boundary conditions. Note that regions 9 and 12 correspond to outflow while in regions 21 and 27 inflow occurs and, therefore, different relations must be applied to calculate the strength of the reflected waves.

It is easily possible to follow the movements of an individual particle. In region 1, the velocity is zero and therefore the particle remains on a line parallel to the time axis until region 2 is reached. There, the velocity is taken from Table 2 and a line of corresponding inclination is drawn. This process is continued and the dotted line represents the path of the particle.

Once the wave diagram is completed Table 2 gives the value of U and A for each region: Other variables as density or pressure may be calculated from the values of A by means of the adiabatic relations. As an example, Fig. 10a shows the variation of pressure p (relative to the pressure p_o outside the tube)at the piston as function of time.

3. Reflection and transmission of a wave at a change of cross-section

Referring to Fig. 5, assume the following

$\frac{S''}{S'} = 1\ 2,\qquad A_1 = 0.980$

$U_1 = 0.400\qquad \Delta A = 0.020$

It has been found very convenient to solve this type of problem by preparing in advance sheets with the general layout of the calculation so that only the numerical values had to be entered. This chart is shown as Fig. 11. The top line contains all the quantities which must be known in advance. The next three lines contain quantities calculated in the straight forward way shown in Section III. Then, a guess for $\Delta A''$ must be made. With the aid of Fig. 7, the first guess is $\Delta A'' = 0.0187$ and the computation proceeds until finally two values for $\Delta A'$ are obtained. They are distinguished by the subscripts A and U depending on whether they were obtained from the difference of the A or U values, respectively. A second guess for $\Delta A''$ was then made which led to the final result $\Delta A' = -0.005$ and $\Delta A'' = 0.019$.

APPENDIX:
Remarks on the Characteristics Theory of Hyperbolic Differential Equations

The following principles and theorems are merely stated without proof to correlate the treatment of waves as shown in Section I with the characteristics theory of hyperbolic partial differential equations[4,11,12]. The principles are similar to those of the widely used method of solving problems in steady, two-dimensional, supersonic flow by Prandtl and Busemann[13].

From the equation of continuity (1) and of motion (2) a single partial differential equation of second order may be derived by introducing a velocity potential defined by $u = \frac{\partial \phi}{\partial x}$ The potential equation becomes

$$\phi_{tt} + 2\phi_x \, \phi_{xt} + (\phi_x{}^2 - a^2)\, \phi_{xx} = 0 \qquad (25)$$

where subscripts denote partial differentiations.

This equation is a special case of the general partial differential equation of second order

$$aZ_{xx} + 2\,b\,Z_{xt} + cZ_{tt} + d = 0$$

which is called elliptic, parabolic or hyperbolic depending on whether $ac-b^2$ is positive, zero or negative. If the coefficients a, b, and c depend on the independent variables x and t alone, the equation is called linear.

For the potential equation (25) the expression $ac-b^2$ becomes

$$(\phi_x^2 - a^2) - \phi_x^2 = -a^2 < 0$$

indicating that it is hyperbolic. Since the coefficients are not functions of x and t only, the equation is also non-linear.

In the case of a hyperbolic equation, there exist two families of real curves in the x, t-plane which are given by

$$a\,dt^2 - 2b \cdot dx\,dt + c\,dx^2 = 0 \tag{26}$$

These curves are called the characteristics or characteristic curves of the differential equation. The following existence theorem can be proven. If the values of the function Z and its first derivatives Z_x and Z_t are given along a line AB (Fig. 12) which must not be a characteristic, then the values of Z are definitely determined in the quadrangle APBQ which is formed by the characteristics through A and B. If the differential equation is linear then the quadrangle APBQ depends on the points A and B only, while in the case of a non-linear differential equation, it also depends on the values of Z and its derivatives along AB.

A variation of the initial values of Z or its derivatives in a segment CB of the line AB will modify the solution only within the region bounded by CRPBQS while no change takes place in the remaining quadrangle ARCS. It thus becomes clear that variations of the initial conditions propagate along characteristic lines.

In general, it is not possible to find a continuous solution of equation (25) which also satisfies the given boundary conditions. However, an approximate step by step solution may be obtained by assuming constant values of ϕ_x and ϕ_t within sufficiently small regions of the x, t-plane. Since a variation of conditions propagates only along characteristic lines as pointed out above, it follows that these regions must be bounded by characteristics. Physically, this method of solving the differential equation means replacing the non-steady flow solution by a large number of small regions of steady flow. Any desired accuracy may be obtained merely by decreasing the size of these regions. Since two characteristic lines pass through each point of the x, t-plane the whole plane is covered by a characteristics network.

Using equation (26), the characteristics of the potential equation (25) are given by

$$(u^2 - a^2)\, dt^2 - 2u\, dx\, dt + dx^2 = 0$$

or $$\frac{dx}{dt} = u \pm a$$

which corresponds to the statement of Section I that values of P and Q propagate with a velocity of (u + a) and (u - a) respectively. Since u and a are not constant but depend on the given problem, the characteristic lines cannot be drawn in advance but must be constructed from the initial and boundary conditions. This is a direct consequence of the fact that the potential equation is non-linear.

It is possible to linearize equation (25) by a change from the independent variables x and t to the new variables u and a. By this process, ϕ is transformed into a new variable Ω and the equation becomes[4].

$$\Omega_{uu} - \frac{(\gamma-1)^2}{4}\Omega_{aa} + \frac{(\gamma-1)(\gamma-3)}{4}\frac{\Omega_a}{a} = 0$$

This is a linear hyperbolic equation with a and u as independent variables. The characteristics of this equation are independent of the given problem and defined by

$$\frac{du}{da} = \pm \frac{2}{\gamma-1}$$

This corresponds to equation (6) in Section I which relate the relative values of increments for a and u.

ACKNOWLEDGMENT

The writers wish to thank Miss E. Evans and Miss R. Harbach who worked out the examples and prepared the figures, and Dr. J. V. Foa, Dr. G. Markstein and Mr. J. Logan for valuable suggestions for the preparation of the paper.

BIBLIOGRAPHY

1. Schultz-Grunow, F. Non-stationary, one-dimensional gas motion Forsch. Ing. Wesen 13, No. 3, 125-134 (1942)

2. Schultz-Grunow, F. Gas dynamic investigation of the pulse-jet tube. Parts I and II NACA T.M. No. 1131, (Feb. 1947)

3. MacDonald, J.K.L. A gas dynamical formulation for waves and combustion in pulse-jets AMG-NYU No. 151, (June 1946)

4. Sauer, R. Theory of non-stationary gas flow
   I Laminar steady gas waves
   AAF Trans. No. F-TS-763-RE (August 1946)

   II Plane gas waves with compression shocks
   AAF Trans. No. F-TS-758-RE (September 1946)

   III Laminar flow in tubes of variable cross section
   AAF Trans. No. F-TS-770-RE (1946)

   IV Plane gas flow with friction, heat transfer and temperature differences
   AAF Trans. No. F-TS-949-RE (March 1947)

5. de Haller, P. On a graphical method of gas dynamics R.T.P. Translation 2555.

6. Kahane, A. and Lees, L. Unsteady, one-dimensional flows with heat addition or entropy grandients Project Squid, Technical Memorandum No. P-2 (November 1947)

7. Riemann, B. On the propagation of plane gas waves of finite amplitude Gött. Abh. 8, p. 43 (1858)

8. Liepmann, H.W. and Puckett, A.E. Introduction to aerodynamics of a compressible fluid New York, Wiley (1947)

9. Ackeret, J. Gas dynamics. R.T.P. Translation No. 2119

10. De Prima, C. and Leopold, A. On the effectiveness of various modes of detonation or combustion AMG-NYU No. 145, February 1946

11. Courant, R. and Hilbert, D. Methoden der mathematischen Physik Vol. II Berlin, Springer, (1937)

12. Preiswerk, E. Application of the methods of gas dynamics to water flows with free surface, Part I. NACA T.M. No. 934 (March 1940)

13. Prandtl, L. and Busemann, A. Approximation method for the graphical determination of two-dimensional supersonic flow. Stodola Festschrift, Zürich and Leipzig Orell Füssli (1929) p. 499-509, (Translated by C.A.L.)

TABLE I

$$K = \left(\frac{A}{A^*}\right)^{2\beta} \cdot \left[1 - \left(\frac{A}{A^*}\right)^2\right].$$

$\gamma = 1.400 \qquad \beta = 5$

$\frac{A}{A^*}$	K	$\frac{A}{A^*}$	K	$\frac{A}{A^*}$	K	$\frac{A}{A^*}$	K
.901	.0664	.926	.0661	.951	.0578	.976	.0372
.902	.0664	.927	.0659	.952	.0575	.977	.0360
.903	.0665	.928	.0658	.953	.0567	.978	.0348
.904	.0666	.929	.0656	.954	.0561	.979	.0336
.905	.0667	.930	.0654	.955	.0555	.980	.0324
.906	.0668	.931	.0652	.956	.0549	.981	.0311
.907	.0668	.932	.0650	.957	.0542	.982	.0297
.908	.0669	.933	.0647	.958	.0535	.983	.0284
.909	.0669	.934	.0645	.959	.0528	.984	.0270
.910	.0669	.935	.0642	.960	.0521	.985	.0256
.911	.0670	.936	.0639	.961	.0514	.986	.0241
.912	.0670	.937	.0636	.962	.0506	.987	.0227
.913	.0670	.938	.0634	.963	.0498	.988	.0211
.914	.0670	.939	.0630	.964	.0490	.989	.0196
.915	.0670	.940	.0627	.965	.0482	.990	.0180
.916	.0670	.941	.0623	.966	.0473	.991	.0164
.917	.0669	.942	.0620	.967	.0464	.992	.0147
.918	.0668	.943	.0616	.968	.0455	.993	.0130
.919	.0668	.944	.0612	.969	.0446	.994	.0113
.920	.0667	.945	.0608	.970	.0436	.995	.0095
.921	.0666	.946	.0603	.971	.0426	.996	.0077
.922	.0666	.947	.0599	.972	.0416	.997	.0058
.923	.0664	.948	.0594	.973	.0405	.998	.0039
.924	.0663	.949	.0589	.974	.0394	.999	.0020
.925	.0662	.950	.0584	.975	.0383	1.000	.0000

TABLE 2

$$\vec{W} = U \pm \left[A + 3\ \Delta A + 3.5\ \frac{(\Delta A)^2}{A} \right]$$ for compression waves ($A > 0$)

$W = U \pm A$ for expansion waves ($A < 0$)

$\vec{W}$ Wave velocity from left to right $\overleftarrow{W}$ Wave velocity from right to left

Region	U	A	$\vec{W}$	$\overleftarrow{W}$	Region	U	A	$\vec{W}$	$\overleftarrow{W}$
1	0.000	1.000	1.126		27	-.340	0.988	0.774	
2	0.200	1.040	1.240	-0.840	28	.100	0.980	1.080	-1.080
3	0.100	1.020	1.120	-0.920	29	.200	0.960	1.160	-0.886
4	0.000	1.000	1.000	-1.000	30	.000	1.000	1.000	-1.051
5	-0.100	0.980	0.880	-1.080	31	-.085	1.017	0.932	-1.138
6	-0.200	0.960	0.886	-1.160	32	-.140	1.028	0.888	-1.168
7	0.000	1.000		-1.000	33	.000	0.960		-0.960
8	0.300	1.020	1.320	-0.720	34	.100	0.940	1.040	-0.966
9	0.400	1.000	1.400		35	.000	0.920		-1.046
10	0.200	1.000	1.200	-0.800	36	-.040	1.008	0.968	-1.048
11	0.300	0.980	1.280	-0.740	37	-.100	0.980	0.880	-1.131
12	0.200	1.000	1.200		38	-.185	0.997	0.812	-1.215
13	0.100	0.980	1.080	-0.880	39	-.240	1.008	0.768	-1.248
14	0.200	0.960	1.160	-0.820	40	.000	1.000	1.000	
15	0.100	0.980	1.080	-0.940	41	-.200	0.960	0.886	-1.211
16	0.000	1.000	1.000		42	.000	1.000		
17	0.000	0.960	0.960	-0.960	43	-.285	0.977	0.818	-1.295
18	0.100	0.940	1.040	-0.900	44	-.140	0.988	0.848	-1.128
19	0.000	0.960	0.960	-1.020	45	-.100	0.980	0.880	-1.131
20	-0.100	0.980	0.880	-1.131	46	-.185	0.997	0.812	
21	-0.185	0.997	0.812		47	-.340	0.988	0.774	-1.328
22	-0.100	0.940	0.966	-1.040	48	-.240	0.968	0.854	-1.208
23	0.000	0.920	1.046	-0.980	49	-.200	0.960	0.886	-1.211
24	-0.100	0.940	0.846	-1.100	50	-.285	0.977	0.818	-1.295
25	-0.200	0.960	0.886	-1.211	51	-.340	0.988	0.774	
26	-0.285	0.977	0.818	-1.295					

APPROXIMATION OF A WAVE BY FINITE STEPS

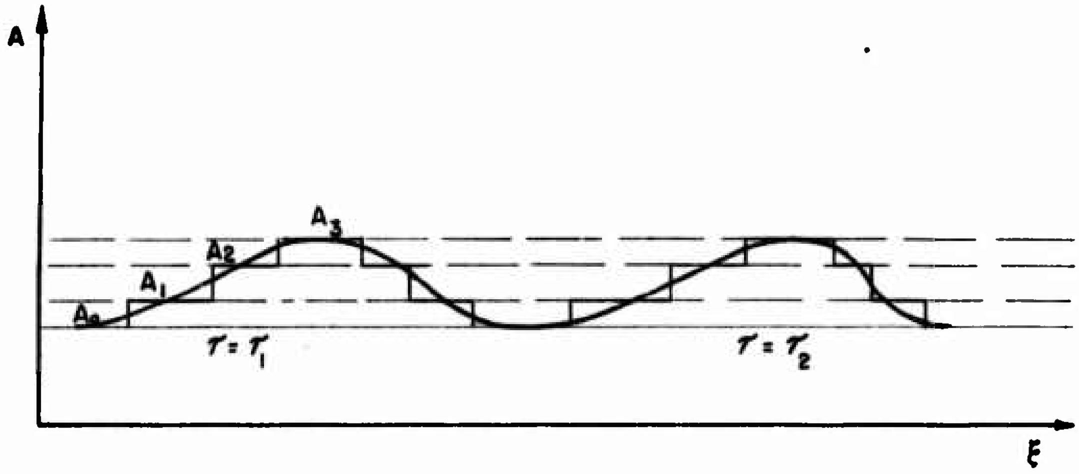

Fig. 1

REPRESENTATION OF WAVE ELEMENTS IN THE ξ, τ - PLANE. INSERT: CONSTRUCTION OF A WAVE OF GIVEN VELOCITY <u>W</u>

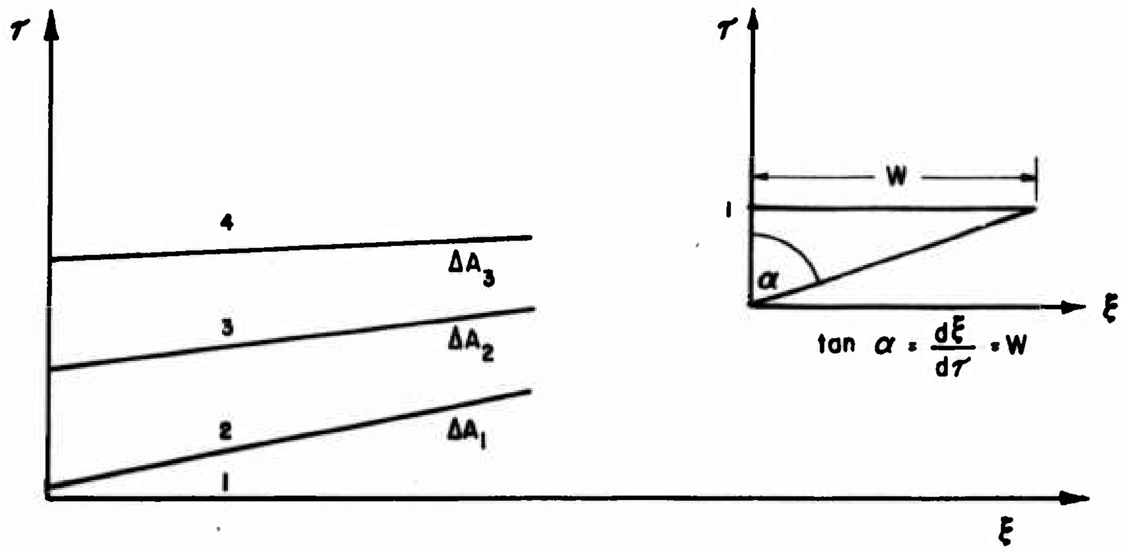

Fig. 2

INTERSECTION OF TWO WAVE ELEMENTS

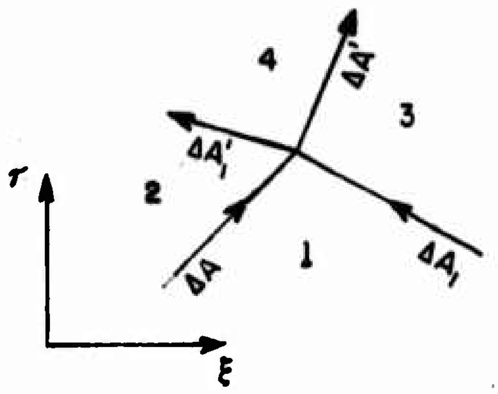

Fig. 3

WAVE REFLECTIONS AT THE END OF A TUBE

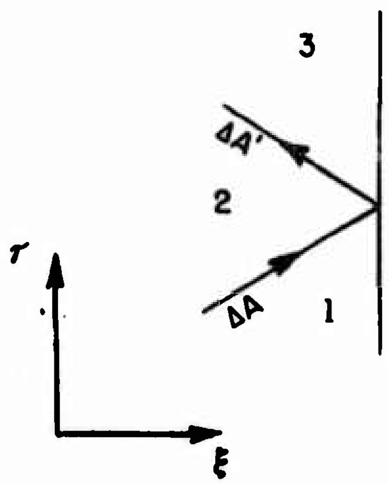

Fig. 4

WAVE REFLECTION AT A CHANGE OF CROSS SECTION OF A TUBE

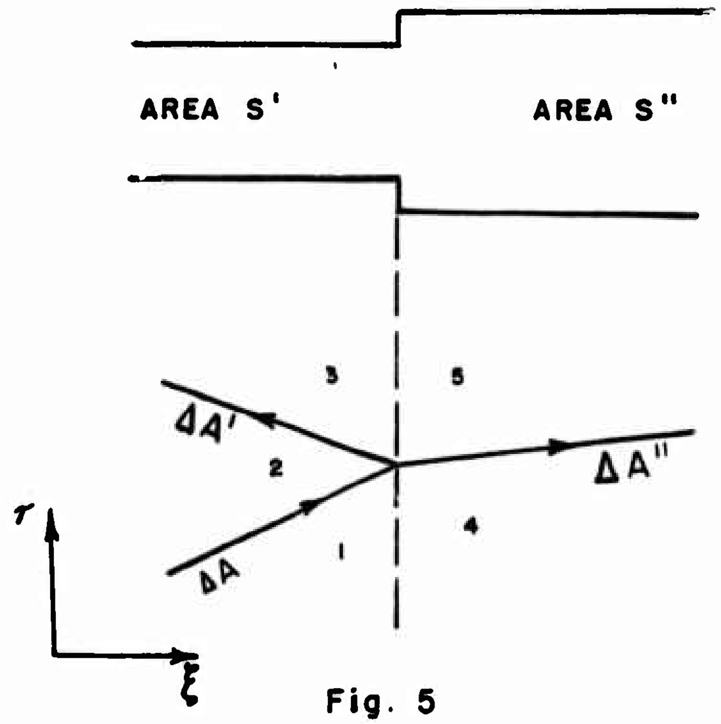

Fig. 5

PLOT OF K AS FUNCTION OF A/A* (EQUATION 18)

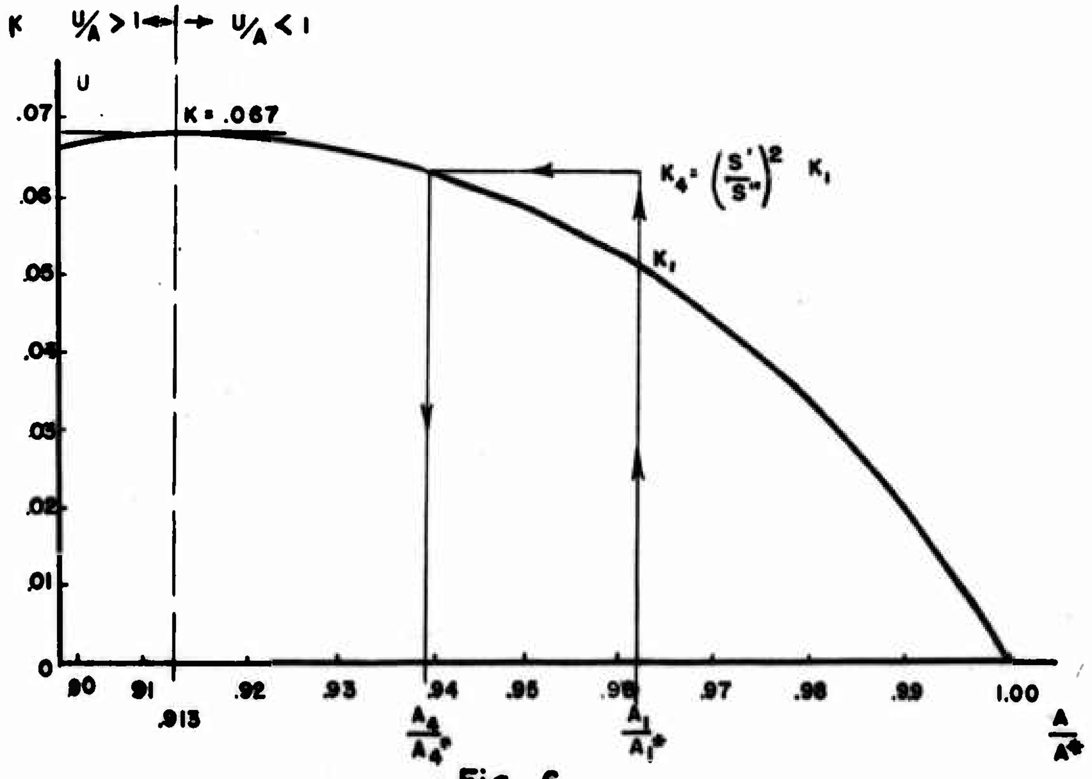

Fig. 6

AID FOR MAKING A FIRST GUESS OF $\Delta A''$

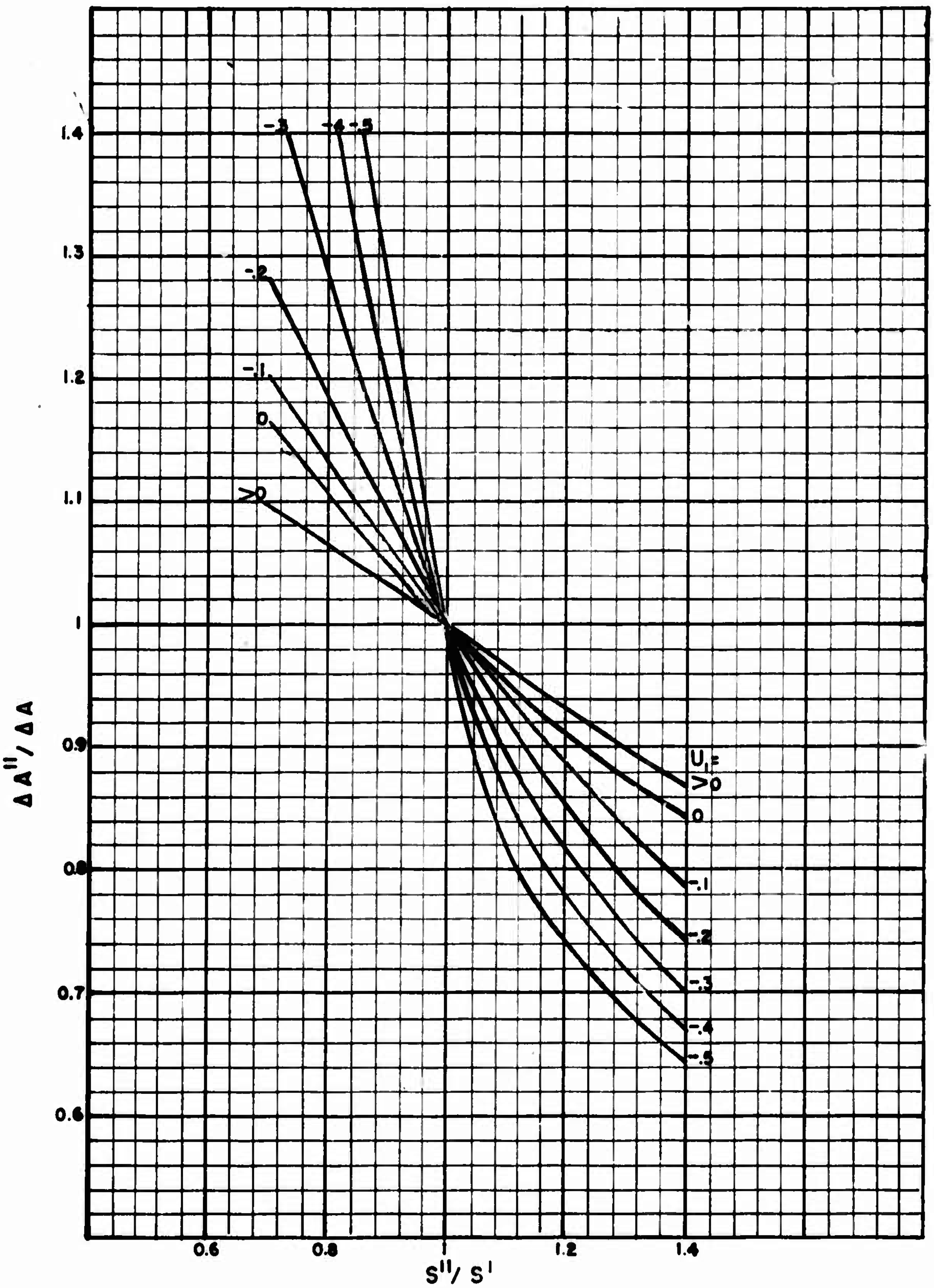

Fig. 7

WAVE REFLECTION AT AN INTERFACE

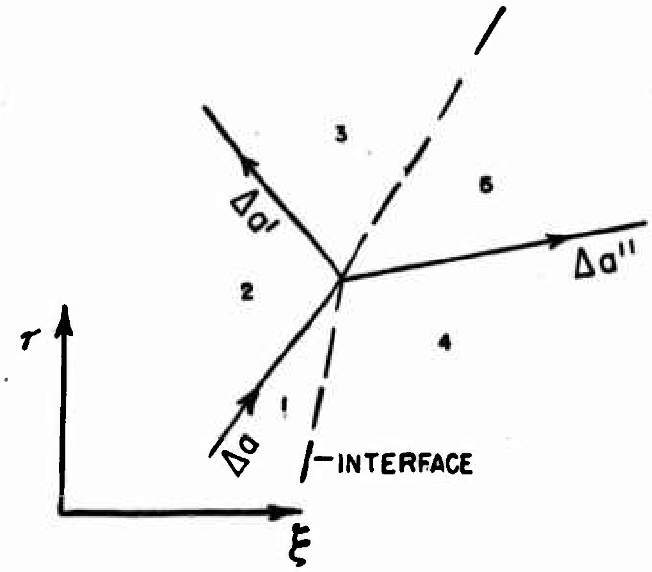

FIG. 8

CREATION OF AN INTERFACE BY REMOVAL OF A PARTITION BETWEEN TWO GASES

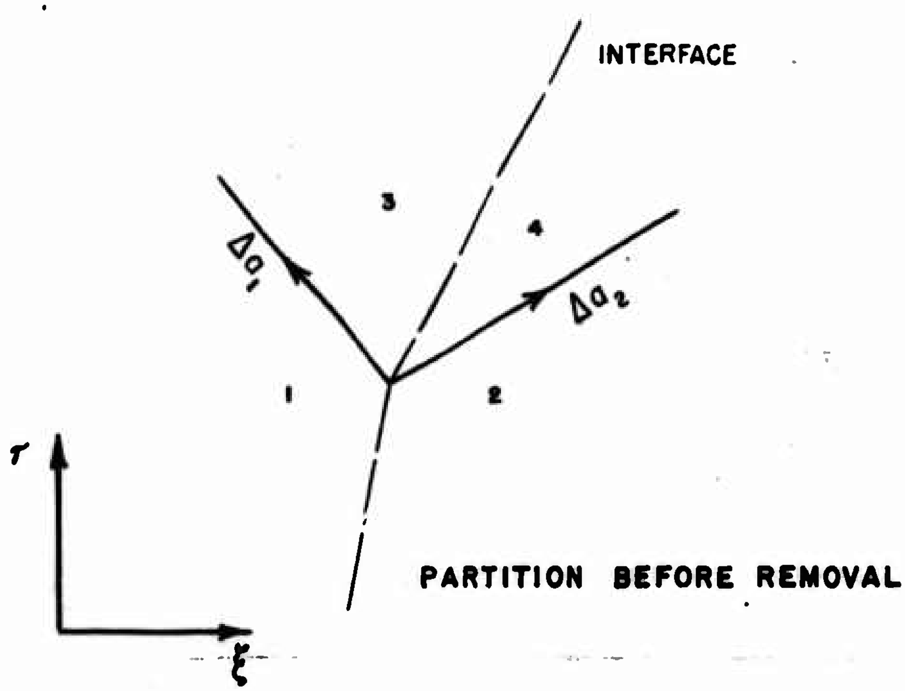

Fig. 9

WAVE DIAGRAM FOR THE EXAMPLE OF A PISTON MOVING IN AN OPEN TUBE

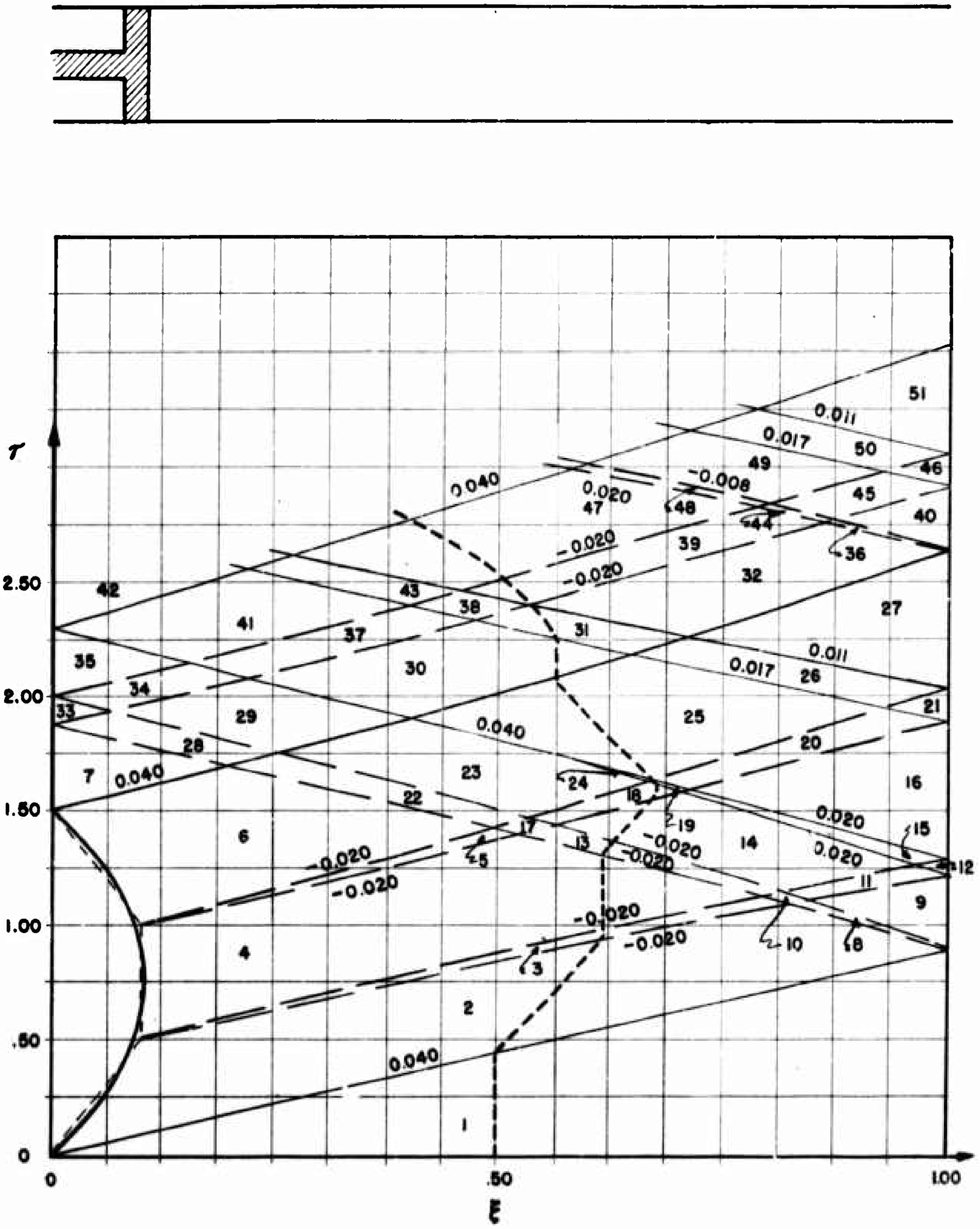

Fig. 10

WAVE DIAGRAM FOR THE EXAMPLE OF A PISTON MOVING IN AN OPEN TUBE

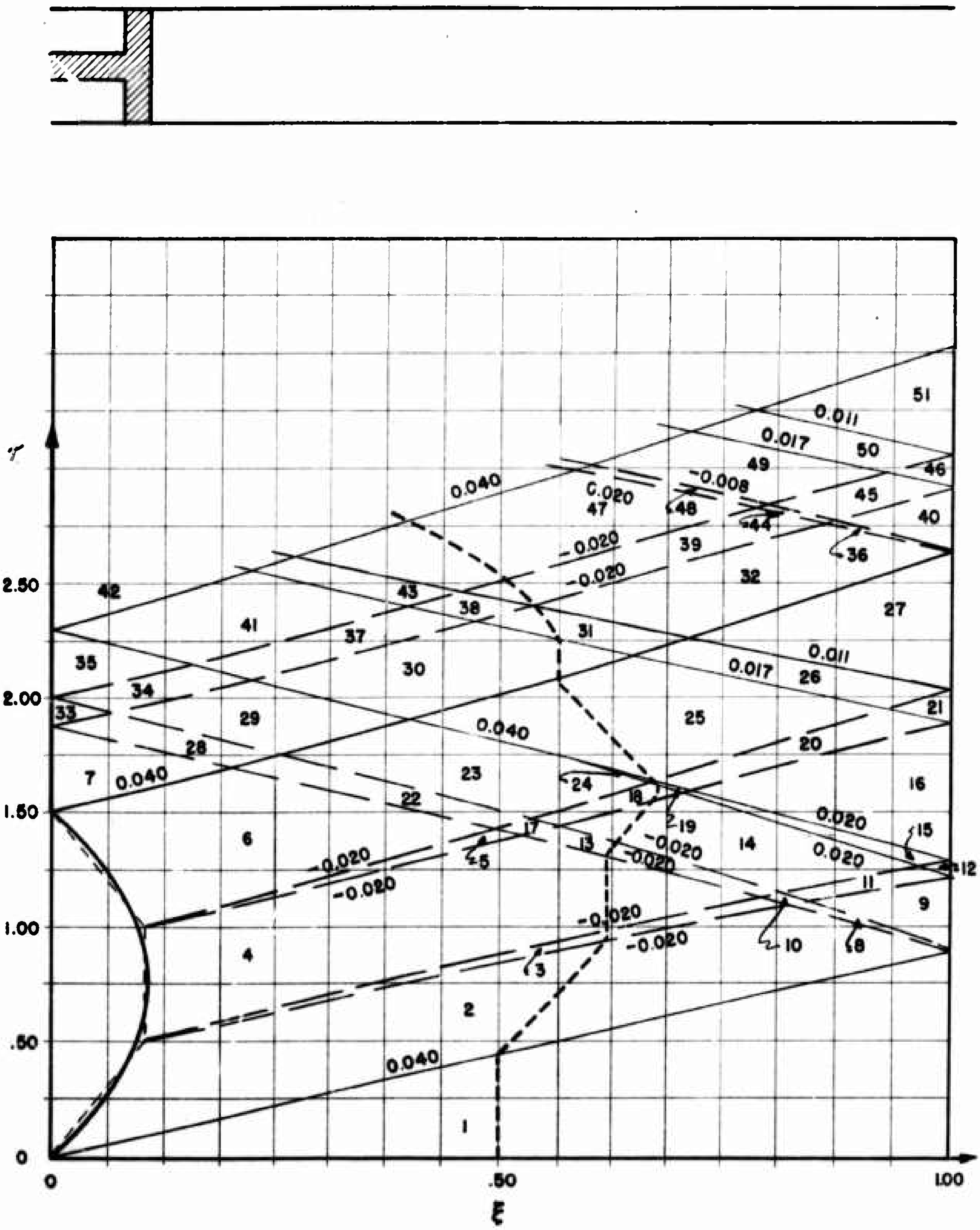

Fig. 10

RELATIVE PRESSURE P/P_o AT THE PISTON AS FUNCTION OF TIME

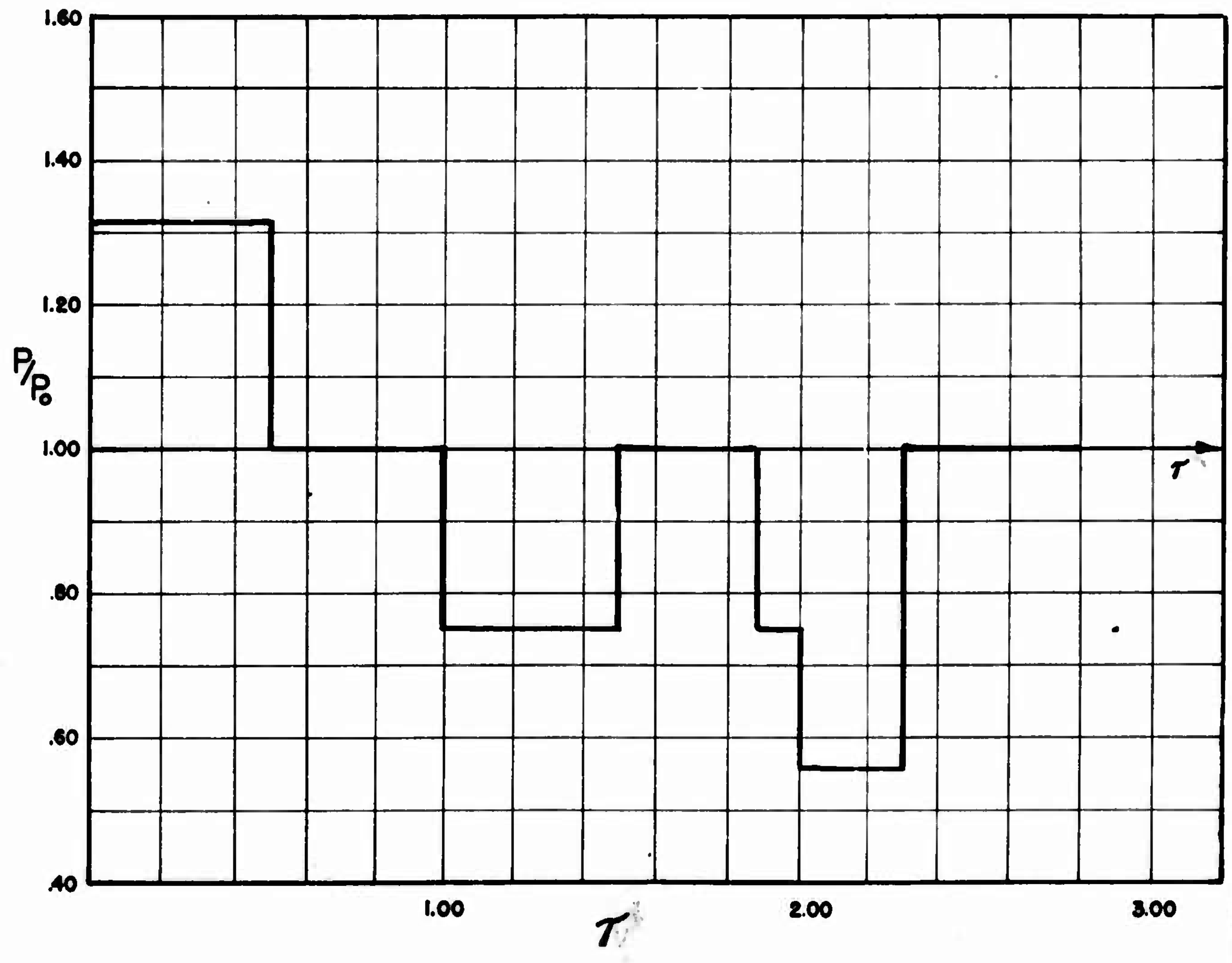

Fig. 10a

CALCULATION FOR WAVE REFLECTION AT A CHANGE OF CROSS SECTION

$\frac{S''}{S'} = 1.2$	$A_1 = 0.980$	$U_1 = 0.400$	$\Delta A = 0.020$
$\left(\frac{S''}{S'}\right)^2 = 1.44$	$\frac{U_1}{A_1} = 0.4082$	$A_2 = 1.000$	$U_2 = 0.5000$
$\frac{A_1}{A_1^*} = 0.9838$	$K_1 = 0.0273$	$K_4 = 0.0189$	$\frac{A_4}{A_1^*} = 0.9895$
$A_1^* = 0.9961$	$\frac{U_4}{A_4} = 0.3250$	$A_4 = 0.9856$	$U_4 = 0.3203$
$\Delta A'' = 0.0187$	$A_5 = 1.0043$	$U_5 = 0.4138$	$\frac{U_5}{A_5} = 0.4120$
$\frac{A_5}{A_5^*} = 0.9835$	$K_5 = 0.0277$	$K_3 = 0.0399$	$\frac{A_3}{A_5^*} = 0.9736$
$A_5^* = 1.0211$	$\frac{U_3}{A_3} = 0.5240$	$A_3 = 0.9941$	$U_3 = 0.5209$
$\Delta A'_A = A_3 - A_2 = -0.0059$		$\Delta A'_U = \frac{U_2 - U_3}{\beta} = -0.0042$	
$\Delta A'' = 0.0190$	$A_5 = 1.0046$	$U_5 = 0.4153$	$\frac{U_5}{A_5} = 0.4134$
$\frac{A_5}{A_5^*} = 0.9833$	$K_5 = 0.0280$	$K_3 = 0.403$	$\frac{A_3}{A_5^*} = 0.9732$
$A_5^* = 1.0217$	$\frac{U_3}{A_3} = 0.5280$	$A_3 = 0.9943$	$U_3 = 0.5250$
$\Delta A'_A = A_3 - A_2 = -0.0057$		$\Delta A'_U = \frac{U_2 - U_3}{\beta} = -0.0042$	
	Result: $\underline{\underline{\Delta A' = -0.005}}$	$\underline{\underline{\Delta A'' = 0.019}}$	

Fig. 11

ILLUSTRATION FOR THE EXISTENCE THEOREM OF THE CHARACTERISTICS THEORY

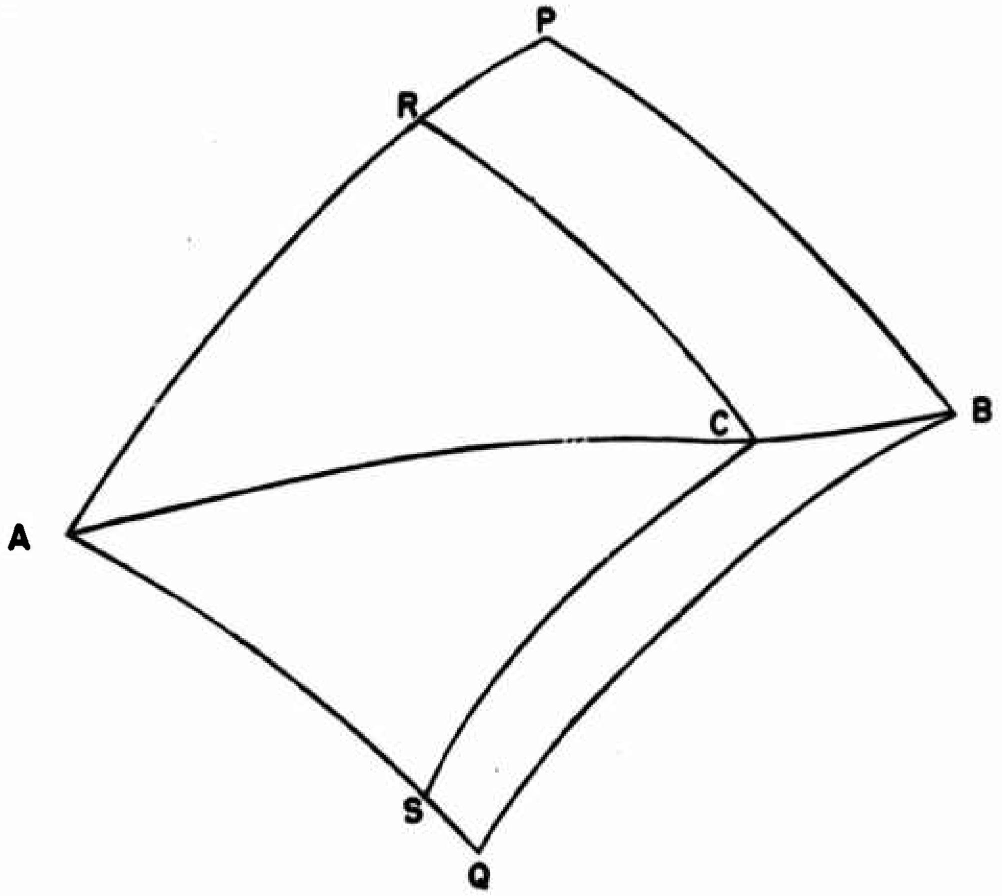

Fig. 12